CHAPTER 11

LIMITS, L'HÔPITAL'S RULE, AND NUMERICAL METHODS

11.1 Limits of Functions

PREREQUISITES

1. Recall the basic properties of limits (Section 1.2).
2. Recall the concept of horizontal and vertical asymptotes (Sections 1.2 and 3.4).
3. Recall how to compute limits at infinity or infinite limits (Section 1.2).

PREREQUISITE QUIZ

1. Complete the following statements (assuming all limits are finite):
   (a) $\lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = \lim_{x\to x_0}$ _____ .
   (b) $\lim_{x\to x_0} 3g(x) = [\lim_{x\to x_0} g(x)] \times$ _____ .
   (c) $\lim_{x\to\infty}[f(x)/g(x)] =$ _____ / $\lim_{x\to\infty} g(x)$, provided $\lim_{x\to\infty} g(x) \neq 0$.
2. What is $\lim_{x\to 0}(1/x)$? Explain your answer.
3. Does $\lim_{x\to\infty}[(x + 1)/x^2]$ exist? If yes, what is it?
4. Sketch the graph of $y = x^2/(1 - x^2)$. Discuss the asymptotes.

GOALS

1. Be able to state the $\varepsilon - \delta$ definition of the limit and apply it to prove basic general properties of limits.

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2. Be able to evaluate simple specific limits.

3. Be able to compute limits at infinity.

4. Be able to compute one-sided limits.

STUDY HINTS

1. Limit definition. The $\varepsilon - \delta$ definition of the limit is not useful for computing limits. It is, however, very important in theoretical work. Example I demonstrates how the definition is used. We write $f(x) - \ell$ in terms of $x - x_0$. This will permit us to find a relationship between δ and ε so that $|x - x_0| < \delta$ ensures that $|f(x) - \ell| < \varepsilon$. Remember that ε is given and your job is to find a δ which fits the definition. Both ε and δ are positive numbers.

2. Basic properties. The box on p. 511 was already presented in Chapter 1. Again, you are reminded that "common sense" should help you recall all of the properties.

3. Important methods. Example 3 shows two important tricks used in limit computations. If the denominator at x_0 is zero, try to factor out $x - x_0$. A second trick is rationalization. If a radical appears in the denominator, use $a^2 - b^2 = (a - b)(a + b)$ to eliminate the radical. Also $a^3 - b^3 = (a - b)(a^2 + 2ab + b^2)$ may be used in some situations.

4. Limits at infinity. To compute limits, divide numerator and denominator by the largest degree of x in the numerator. Then use the fact that $\lim_{x\to\infty}(1/x) = 0$. A horizontal asymptote, if it exists, is simply the limit at $-\infty$ or $+\infty$.

5. Infinite limits. These have already been discussed when vertical asymptotes were introduced in Chapter 3.

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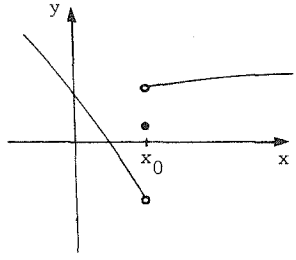
6. Reciprocal test for infinite limits. If the limit of the reciprocal is zero, then the limit of the original function may be $+\infty$, $-\infty$, or neither. Which occurs depends on how $f(x)$ behaves about x_0. If the sign is positive on one side of x_0 and negative on the other side, then no limit exists.

7. 

One-sided limits. The definition is just like that for ordinary limits except that we now consider the intervals $(x_0, x_0 + \delta)$ or $(x_0 - \delta, x_0)$ rather than $(x_0 - \delta, x_0 + \delta)$. The concept of one-sided limits is important when the value of a function changes from $-\infty$ to $+\infty$. This is also important when absolute values are involved. The figure demonstrates that one-sided limits may exist even though the (two-sided) limit does not exist at x_0.

8. Comparison test. Thinking of this as the "sandwich principle" makes the test simple to understand. You should understand why the nickname is appropriate.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $\varepsilon > 0$ be given. We have to find $\delta > 0$ such that $|x^2 - a^2| < \varepsilon$ if $|x - a| < \delta$. We note that $x^2 - a^2 = (x - a)^2 + 2a(x - a)$. If $|x - a| < \delta$, we have $|x^2 - a^2| = |(x - a)^2 + 2a(x - a)| \leq |x - a|^2 + 2|a||x - a| < \delta^2 + 2|a|\delta$. Thus, we want δ to be such that $\delta^2 + 2|a|\delta \leq \varepsilon$. We may choose δ such that $\delta \leq 1$ (so $\delta^2 \leq \delta$) and $\delta \leq \varepsilon/(1 + 2|a|)$. This choice (or a smaller one) guarantees that $\delta^2 + 2|a|\delta \leq (1 + 2|a|)\delta \leq \varepsilon$, and so $|x^2 - a^2| < \varepsilon$ if $|x - a| < \delta$.

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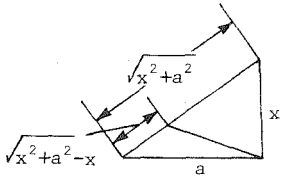
5. Note that the exponential function is continuous at 0, so $\lim_{\theta\to 0} \exp[(3\tan\theta)/\theta] = \exp\lim_{\theta\to 0}[3\tan\theta/\theta] = \exp[3\lim_{\theta\to 0}(\tan\theta/\theta)] = e^3$.

9. Using the replacement rule, we get $\lim_{x\to 2}[(x^2 - 4)/(x^2 - 5x + 6)] = \lim_{x\to 2}[(x - 2)(x + 2)/(x - 2)(x - 3)] = \lim_{x\to 2}[(x + 2)/(x - 3)] = -4$.

13. We need to find A such that $|f(x) - \ell| < \varepsilon$ whenever $x > A$. We have $|f(x) - \ell| = |(1 + x^3)/x^3 - 1| = |(1 + x^3 - x^3)/x^3| = |1/x^3|$. To make this less than ε, we note that $1/x^3 < \varepsilon$ whenever $x > 1/\sqrt[3]{\varepsilon}$, so we may choose $A = 1/\sqrt[3]{\varepsilon}$.

17. Since $\lim_{x\to\infty}(1/x) = 0$, $\lim_{x\to\infty}(3/x + 5/x^2 - 2) = 3\lim_{x\to\infty}(1/x) + 5\left[\lim_{x\to\infty}(1/x)\right] \times \lim_{x\to\infty}(1/x) - 2\lim_{x\to\infty}(1) = 0 + 0 - 2 = -2$.

21. Divide through by x^2/x^2 to get $\lim_{x\to\infty}[(3x^2 + 2x + 4)/(5x^2 + x + 7)] = \lim_{x\to\infty}[(3 + 2/x + 4/x^2)/(5 + 1/x + 7/x^2)]$. Using the fact that $\lim_{x\to\infty}(1/x) = 0$ and the other rules for limits, this becomes $(3 + 0 + 0)/(5 + 0 + 0) = 3/5$.

25. 

Upon rationalizing, we have $\lim_{x\to\infty}(\sqrt{x^2 + a^2} - x) = \lim_{x\to\infty}[(\sqrt{x^2 + a^2} - x)(\sqrt{x^2 + a^2} + x)/(\sqrt{x^2 + a^2} + x)] = \lim_{x\to\infty}[(x^2 + a^2 - x^2)/(\sqrt{x^2 + a^2} + x)] = a^2\lim_{x\to\infty}[1/(\sqrt{x^2 + a^2} + x^2)] = 0$. Geometrically, we can look at $\sqrt{x^2 + a^2} - x$ as the difference between the hypotenuse and one leg of a right triangle. As x gets larger and larger with a fixed, the difference is getting smaller and smaller.

29. We have $\lim_{x\to 2}(x - 2)^2 = 0$ and $1/(x - 2)^2 \geqslant 0$ for all x. Thus, by the reciprocal test, $\lim_{x\to 2}[1/(x - 2)^2] = +\infty$.

33. Factoring out $x - 2$ yields $\lim_{x\to 2+}[(x + 2)/(x - 2)]$. This approaches $4/0$; however, $x - 2 > 0$ for $x > 2$. Therefore, the limit is $+\infty$.

37. Note that $|x| > 0$ and $x > 0$ as $x \to 0+$. Thus, $\lim_{x\to 0+}(|x|/x) = 1$ and $\lim_{x\to 0+}[(x^3 - 1)|x|/x] = \lim_{x\to 0+}(x^3 - 1) = -1$.

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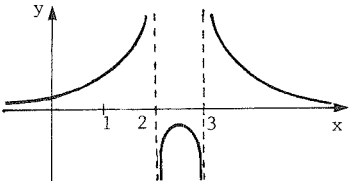
41. 

Vertical asymptotes occur where $\lim_{x \to x_0} f(x) = \pm\infty$ and horizontal asymptotes occur at $y = \lim_{x \to \pm\infty} f(x)$. We find that $f(x) = 1/(x^2 - 5x + 6) = 1/(x - 3)(x - 2)$. Thus, $f(x) \to \pm\infty$ as $x \to 3\pm$ and $f(x) \to \mp\infty$ as $x \to 2\pm$, so $x = 3$ and $x = 2$ are vertical asymptotes. Also, $\lim_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is the horizontal asymptote.

45. (a) Given $\varepsilon > 0$, there is an $A > 0$ such that $|f(x)| < \varepsilon$ whenever $x > A$ by the assumption that $\lim_{x \to \infty} f(x) = 0$. Given $\varepsilon > 0$, this same δ also gives $|g(x)| < \varepsilon$ whenever $x > A$ since $|g(x)| < |f(x)|$. Hence, g has limit 0 as $x \to \infty$, as well.

(b) Let $g(x) = \sin(1/x)/x$ and $f(x) = 1/x$. Then $|g(x)| \leq |f(x)|$ and the sandwich principle applies. $\lim_{x \to \infty} f(x) = 0$, so $\lim_{x \to \infty} g(x) = 0$ also.

49. The numerator is $(x - 1)(x^2 + x + 1)$ and the denominator is $(x - 1) \times (x + 1)$. Therefore, if $x^2 - 1 \neq 0$, then $\lim_{x \to 1}[(x^3 - 1)/(x^2 - 1)] = \lim_{x \to 1}[(x^2 + x + 1)/(x + 1)] = 3/2$.

53. If $x \neq -1$, then long division shows us that $\lim_{x \to -1}[(x^{2n+1} + 1)/(x + 1)] = \lim_{x \to -1}(x^{2n} - x^{2n-1} + x^{2n-2} - x^{2n-3} + \ldots + x^4 - x^3 + x^2 - x + 1)$. The limit of the terms with even exponents is 1 , and the limit of the terms with odd exponents is $-(-1) = 1$. There are $2n + 1$ terms, so the answer is $2n + 1$.

57. The numerator approaches $e - 1$, which is finite and nonzero, while the denominator approaches 0 on the positive side; therefore, the limit is $+\infty$.

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61. By definition, when $\lim_{x\to\infty} f(x) = \ell$ or $\lim_{x\to-\infty} f(x) = \ell$, the line $y = \ell$ is called a horizontal asymptote. A vertical asymptote occurs where a one-sided limit of $f(x)$ at x_0 is $+\infty$ or $-\infty$. Vertical asymptotes usually occur where the denominator is 0. Division by x^2/x^2 yields $y = (1/x)/(1 - 1/x^2)$, so $\lim_{x\to\infty} f(x) = 0$, which is the horizontal asymptote. The vertical asymptotes are $x = -1$ and $x = 1$.

65. Let $f(x) = \Sigma_{i=0}^{N} a_i x^i$ and let $g(x) = \Sigma_{i=0}^{M} b_i x^i$. If $N < M$, then we can divide by x^M/x^M to get $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to\infty}[a_N x^N/x^M + a_{N-1}x^{N-1}/x^M + \ldots + a_0/x^M] / \lim_{x\to\infty}[b_M + b_{M-1}/x + b_{M-2}/x^2 + \ldots + b_0/x^M]$. Since $N < M$, the limit of the numerator is 0 because $\lim_{x\to\infty}(1/x) = 0$. The limit of the denominator is b_M, so $\lim_{x\to\infty}[f(x)/g(x)] = 0$ if $N < M$. We also know that $\lim_{x\to-\infty}(1/x) = 0$, so $\lim_{x\to-\infty}[f(x)/g(x)] = 0/b_M = 0$ by a similar method. If $N = M$, then we can divide by x^N/x^M to get $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to\infty}[a_N + a_{N-1}/x + a_{N-2}/x^2 + \ldots + a_0/x^N] / \lim_{x\to\infty}[b_M + b_{M-1}/x + b_{M-2}/x^2 + \ldots + b_0/x^M]$. Since $\lim_{x\to\infty}(1/x) = 0$, we have $\lim_{x\to\infty}[f(x)/g(x)] = a_N/b_M$. By a similar method, $\lim_{x\to-\infty}[f(x)/g(x)] = a_N/b_M$. If $N > M$, then we can divide by x^M/x^M to get $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to\infty}[a_N x^N/x^M + a_{N-1}x^{N-1}/x^M + \ldots + a_0/x^M] / \lim_{x\to\infty}[b_M + b_{M-1}/x + b_{M-2}/x^2 + \ldots + b_0/x^M]$. The limit of the denominator is b_M. The numerator is $\lim_{x\to\infty} a_N x^{N-M}$. If $a_N < 0$, then $\lim_{x\to\infty}[f(x)/g(x)] = -\infty$ and the limit is $+\infty$ if $a_N > 0$. $\lim_{x\to\infty}[f(x)/g(x)]$ depends on $N - M$. If $N - M$ is even, then $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to-\infty}[f(x)/g(x)]$. If $N - M$ is odd, then $\lim_{x\to\infty}[f(x)/g(x)] = -\lim_{x\to-\infty}[f(x)/g(x)]$.

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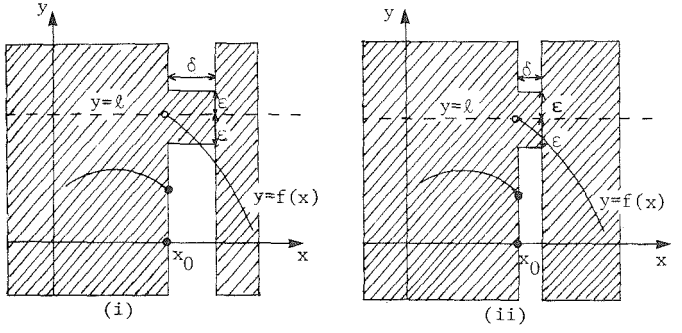
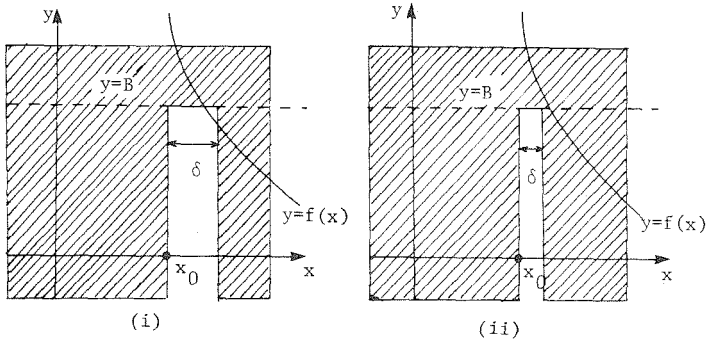
69. (a) To satisfy the definition, the graph of $f(x)$ must be between $\ell - \varepsilon$ and $\ell + \varepsilon$ in the interval $[x_0, x_0 + \delta]$. (i) is an example of a poorly chosen δ. (ii) shows an example of a well-chosen δ.



(b) The definition of the statement requires the graph of $f(x)$ to lie above B in the interval $[x_0, x_0 + \delta]$. (i) shows an example of a poorly chosen δ. (ii) shows an example of a well-chosen δ.



73. Example 5 tells us that $\lim_{x\to\infty} e^{-x} = 0$, so $\lim_{t\to\infty} e^{-\mu t} = 0$ for $\mu > 0$. For fixed x, $\lim_{t\to\infty} T(x,t) = [\lim_{t\to\infty} B_1 \sin\lambda_1 x][\lim_{t\to\infty} e^{-\mu_1 t}] + [\lim_{t\to\infty} B_2 \sin\lambda_2 x] \times [\lim_{t\to\infty} e^{-\mu_2 t}] + [\lim_{t\to\infty} B_3 \sin\lambda_3 x][\lim_{t\to\infty} e^{-\mu_3 t}]$. Since B, λ, and x are

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73. (continued)

constants, the limit is $(B_1\sin\lambda_1 x)(0) + (B_2\sin\lambda_2 x)(0) + (B_3\sin\lambda_3 x)(0) = 0$

77. The composite function rule states that if g is continuous at $\lim_{x\to x_0} f(x)$, then $\lim_{x\to x_0} g(f(x)) = g\left\{\lim_{x\to x_0} f(x)\right\}$. Let $\lim_{x\to x_0} f(x) = L$ and $\varepsilon > 0$ be given. We need to find δ such that $|g(f(x)) - g(L)| < \varepsilon$ whenever $|x - x_0| < \delta$. Since g is continuous at L, there is a $\rho > 0$ such that, whenever $|y - L| < \rho$, $g(y)$ is defined and $|g(y) - g(L)| < \varepsilon$. By the definition of $\lim_{x\to x_0} f(x) = L$, we can find δ such that, whenever $|x - x_0| < \delta$, we also have $|f(x) - L| < \rho$. For such x, we apply the previously obtained property of ρ, with $y = f(x)$, to conclude that $g(f(x))$ is defined and that $|g(f(x)) - g(L)| < \varepsilon$.

81. By the definition of $\lim_{x\to\infty} f(x) = \ell$, we have, whenever $x > A$, $|f(x) - \ell| < \varepsilon$, i.e., $-\varepsilon < f(x) - \ell < \varepsilon$, i.e., $-\varepsilon + \ell < f(x) < \varepsilon + \ell$. Similarly, $\lim_{y\to 0+} f(1/y) = \ell$ means that whenever $0 < y < \delta$, $|f(1/y) - \ell| < \varepsilon$, i.e., $-\varepsilon < f(1/y) < \ell < \varepsilon$, i.e., $-\varepsilon + \ell < f(1/y) < \varepsilon + \ell$. Replacing $1/y$ by x gives $-\varepsilon + \ell < f(x) < \varepsilon + \ell$ whenever $0 < 1/x < \delta$, i.e., $\infty > x > 1/\delta$. This means that $\lim_{x\to\infty} f(x) = \ell$ if $1/\delta = A$. Now, substituting x by $1/y$ gives $-\varepsilon + \ell < f(1/y) < \varepsilon + \ell$ whenever $\infty > 1/y > A$, i.e., $0 < y < 1/A$. This means that $\lim_{x\to 0+} f(1/y) = \ell$ if $1/A = \delta$.

SECTION QUIZ

1. Evaluate the following limits: (State $+\infty$ or $-\infty$, if appropriate.)

   (a) $\lim_{x\to 5+}(|2x - 10|/(x - 5))$

   (b) $\lim_{x\to 5-}(|x - 5|/(2x - 10))$

   (c) $\lim_{t\to -\infty}(|t|/t))$

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1. (d) $\lim_{r\to\infty}[(2r^3 - 5r^2 + 1000)/(3r^3 - 1)]$

   (e) $\lim_{x\to 0}(1/x^5)$

   (f) $\lim_{x\to 0}(1/x^6)$

   (g) $\lim_{x\to 2-}[(x^2 - 5x + 6)/(x - 2)]$

   (h) $\lim_{x\to 3} 1$

   (i) $\lim_{t\to 9}[(81 - t^2)/(3 - \sqrt{t})]$

2. Use the $\varepsilon - \delta$ definition of the limit to prove that $\lim_{x\to 4}(x^3 - x^2) = 48$.

3. Use the $\varepsilon - A$ definition of the limit to prove that $\lim_{x\to\infty}[(x^3 + 1)/x^3] = 1$.

4. City ordinance limits the height of residential buildings to 50 feet. Modern Architect, Inc. wants to build a home whose height is $30 + (20x^3 + x)/(x^3 + x^2 + 2)$ feet , $x \geqslant 0$, where x is the number of centimeters from a designated boundary.

   (a) If a piece of property extends forever, how high will the building be very far from the designated boundary?

   (b) Will Modern Architect violate the building laws? Explain.

   (c) Use the definition of limits to prove your answer in part (a).

ANSWERS TO PREREQUISITE QUIZ

1. (a) $[f(x) + g(x)]$

   (b) 3

   (c) $\lim_{x\to\infty} f(x)$

2. The limit doesn't exist. $\lim_{x\to 0-}(1/x) = -\infty$ and $\lim_{x\to 0+}(1/x) = +\infty$.

3. Yes; zero.

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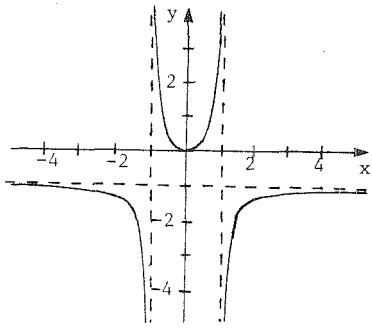
4. 

The horizontal asymptote is $y = -1$.

The vertical asymptotes are $x = -1$ and $x = 1$.

ANSWERS TO SECTION QUIZ

1. (a) 2

   (b) $-1/2$

   (c) -1

   (d) $2/3$

   (e) Does not exist

   (f) $+\infty$

   (g) -1

   (h) 1

   (i) 108

2. Choose $\delta = \sqrt[3]{\varepsilon}$ and show that $|f(x) - L| < \varepsilon$ whenever $|x - 4| < \delta$.

3. Choose $A > \sqrt[3]{1/\varepsilon}$ and show that $|f(x) - L| < \varepsilon$ whenever $x > A$.

4. (a) $\lim_{x\to\infty}[30 + (20x^3 + x)/(x^3 + x^2 + 2)] = 50$

   (b) No, the function is strictly increasing.

   (c) Choose $A > 61/\varepsilon$ and show that $|f(x) - L| < \varepsilon$ whenever $x > A$.

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11.2 L'Hôpital's Rule

PREREQUISITES

1. Recall how to compute a limit (Sections 1.2 and 11.1).
2. Recall the various rules for differentiation (Chapters 1 and 2, Sections 5.2, 5.3, and 6.3).
3. Recall the mean value theorem for differentiation (Section 3.6).

PREREQUISITE QUIZ

1. Let $f(x) = (x - 3)/(x^2 - 5x + 6)$.
   (a) What is $\lim_{x \to 2+} f(x)$?
   (b) What is $-\lim_{x \to 2} |f(x)|$?
   (State $+\infty$ or $-\infty$, if appropriate.)
2. Differentiate $g(x) = (x^2 + e^x)/(3^x - \cos x + \ln x)$.
3. Differentiate $f(x) = \sin(\log_{10} x)$.
4. Let f be differentiable on $[2,3]$, $f(2) = -1$ and $f(3) = 3$. What does the mean value theorem tell you about the slope of the graph on $[2,3]$?

GOALS

1. Be able to know when and how to use l'Hôpital's rule for computing limits of functions.

STUDY HINTS

1. L'Hôpital's rule. This can only be used if the function has the form $0/0$ or ∞/∞ . The process may be repeated as often as necessary, but be sure the function has the appropriate form. You should be concerned about applying this rule, rather than learning its proof, unless you are in an honors class.

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2. Trick number 1. Example 6 shows how to convert a product into a form for which l'Hôpital's rule applies. The trick is to divide by the reciprocal of one of the factors: $uv = u/(1/v)$.

3. Trick number 2. Many times, the limit of an exponential form can be determined by applying l'Hôpital's rule to the logarithm of the desired function. See Example 7. The composite function rule for limits and the continuity of the exponential function permits us to use this trick.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This has the form $0/0$, so $\lim_{x\to 3}[(x^4 - 81)/(x - 3)] = \lim_{x\to 3}(4x^3/1) = 108$.

5. This has the form $0/0$, so $\lim_{x\to 0}[(\cos 3x - 1)/5x^2] = \lim_{x\to 0}(-3 \sin 3x/10x)$ which has the form $0/0$. Applying l'Hôpital's rule again yields $\lim_{x\to 0}(-9 \cos 3x/10) = -9/10$.

9. This has the form ∞/∞ , so $\lim_{x\to\infty}(e^x/x^{375}) = \lim_{x\to\infty}(e^x/375x^{374})$, which again has the form ∞/∞ . Continuing with l'Hôpital's rule yields $\lim_{x\to\infty}(e^x/375\cdot 374x^{373})$. Using l'Hôpital's rule 373 more times gives us $\lim_{x\to\infty}(e^x/375!) = \infty$, where $375! = 375\cdot 374\cdot\ldots\cdot 1$.

13. $x^4 \ln x$ does not have a form to which l'Hôpital's rule applies, but $\lim_{x\to 0}(x^4 \ln x) = \lim_{x\to 0}(\ln x/x^{-4})$, which has the form ∞/∞ . By l'Hôpital's rule, this is $\lim_{x\to 0}[(1/x)/(-4x^{-5})] = \lim_{x\to 0}(-x^4/4) = 0$.

17. We need to transform the function to a form for which l'Hôpital's rule applies. $(\tan x)^x = \exp[x \ln(\tan x)]$ and $x \ln(\tan x) = \ln(\tan x)/(1/x)$ has the form ∞/∞ . Applying l'Hôpital's rule, we have $\lim_{x\to 0}[\ln(\tan x)/(1/x)] = \lim_{x\to 0}[(\sec^2 x/\tan x)/(-x^{-2})] = \lim_{x\to 0}(-x^2/\cos x \sin x)$, which has the form $0/0$. Using l'Hôpital's rule again, we get $\lim_{x\to 0}[-2x/(\cos^2 x - \sin^2 x)] = 0/1$. By the continuous function rule, $\lim_{x\to 0}[(\tan x)^x] = e^0 = 1$.

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21. This has the form $0/0$, so $\lim_{x\to 0}\left[\left(\sqrt{1+x^2}-1\right)/\sin 2x\right] = \lim_{x\to 0}\left[\left(x/\sqrt{1+x^2}\right)/2\cos 2x\right] = 0/2 = 0$.

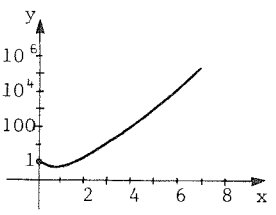
25. This has the form ∞/∞, so $\lim_{x\to\infty}[x/(x^2+1)] = \lim_{x\to\infty}[1/2x] = 0$.

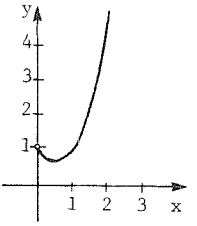
29. This has the form $0/0$, so $\lim_{x\to -1}[(x^2+2x+1)/(x^2-1)] = \lim_{x\to -1}[(2x+2)/2x] = 0/(-2) = 0$.

33. This has the form $0/0$, so $\lim_{x\to\pi}[(1+\cos x)/(x-\pi)] = \lim_{x\to\pi}(-\sin x/1) = 0/1 = 0$.

37. Write $x^p \ln x$ in the form ∞/∞ to use l'Hôpital's rule. $\lim_{x\to 0} x^p \ln x = \lim_{x\to 0}(\ln x/x^{-p}) = \lim_{x\to 0}(x^{-1}/-px^{-p-1}) = \lim_{x\to 0}(x^p/-p) = 0$ because $p > 0$.

41. By Example 7(a), $\lim_{x\to 0+} x^x = 1$. By logarithmic differentiation, we differentiate $\ln y = x \ln x$, so $y'/y = \ln x + 1$. Thus $y' = x^x(\ln x + 1)$, which implies $1/e$ is the critical point. The graph on the left is drawn on a logarithmic scale. The one on the right is drawn on a normal scale.





SECTION QUIZ

1. Use l'Hopital's rule to compute the following limits:

(a) $\lim_{x\to 0+} x(\ln x)^2$

(b) $\lim_{x\to 0+} (1/x)^{(x^2)}$

(c) $\lim_{x\to -\infty} xe^{2x}$

(d) $\lim_{x\to 0}(x^2\cos x/x\sin x)$

(e) $\lim_{x\to 0+}(1/2x - 1/\tan x)$

(f) $\lim_{x\to 0+}(1/x - 1/\tan x)$

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2. The function $f(x) = (\cos x - \sqrt{2}/2)/(x^2 - \pi x/4)$ has the form $0/0$ when $x = \pi/4$. Thus, we apply l'Hôpital's rule to get $\lim_{x\to\pi/4}[(-\sin x)/(2x - \pi/4)] = \lim_{x\to\pi/4}(-\cos x/2) = -\sqrt{2}/4$. This is not correct.

   (a) Why isn't the calculation correct?

   (b) What is the limit $\lim_{x\to\pi/4} f(x)$?

3. Can we use l'Hôpital's rule to find $\lim_{x\to\infty}(\sin x/x)$? Why or why not?

4. Deep in the jungles of Brazil, archaeologists have found a headhunter's recipe for shrinking heads. In addition, the notes of a famous mathematician were found nearby. His observations determined that after t seconds of shrinking, the head was shrunk to x percent of the original size, where x is $100 - (25e^t - t\cos t)/(e^t + 1 - t)$. How effective is the recipe, i.e., how much can a head be shrunk after a very long time?

ANSWERS TO PREREQUISITE QUIZ

1. (a) $+\infty$

   (b) $-\infty$

2. $[(2x + e^x)(3^x - \cos x + \ln x) - (x^2 + e^x)(3^x \ln 3 + \sin x + 1/x)]/(3^x - \cos x + \ln x)^2$

3. $\cos(\log_{10} x)/x \ln 10$

4. Somewhere in $(2,3)$, the slope is 4.

ANSWERS TO SECTION QUIZ

1. (a) 0

   (b) 1

   (c) 0

   (d) 1

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1. (e) $-\infty$

   (f) 0

2. (a) $-\sin x/(2x - \pi/4)$ does not have the form $0/0$ or ∞/∞ .

   (b) $2\sqrt{2}/\pi$.

3. No; $\sin(x)$ does not approach 0 nor ∞ as x approaches ∞ .

4. 75% of the original size.

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11.3 Improper Integrals

PREREQUISITES

1. Recall how to compute limits at infinity (Section 1.2).
2. Recall how to compute infinite limits (Section 1.2).
3. Recall the techniques of integration (Chapter 7, Sections 10.1 and 10.2).

PREREQUISITE QUIZ

1. Compute the following limits: (State $\pm\infty$ if appropriate.)
   (a) $\lim_{x\to 1} |1/(x-1)|$
   (b) $\lim_{x\to\infty} [x^2/(x-1)]$
   (c) $\lim_{x\to 5^-} [(5+x)/(25-x^2)]$
2. Evaluate the following integrals:
   (a) $\int (dx/x)$
   (b) $\int (\cos x + x^2)dx$
   (c) $\int xe^x dx$

GOALS

1. Be able to demonstrate the convergence or divergence of improper integrals, and evaluate them.

STUDY HINTS

1. Improper integrals. Integrals may be improper because the integrand approaches $\pm\infty$ for some x_0 or the interval goes to $\pm\infty$. Note that if an integral is the sum of several improper integrals, all of them must separately converge for the entire integral to converge.
2. Comparison test. The important part which needs to be compared is the region near $\pm\infty$. As long as $f(x)$ is finite in the interval $[a,x_0]$, we only need to apply the comparison test on the interval $[x_0,\infty)$ to determine convergence.

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3. Infinite at midinterval. $\int_{-3}^{1}(1/x^2)dx$ may appear to be proper since the integrand is finite at the endpoints. However, you should notice that the integral becomes improper at $x_0 = 0$. You should always look for points where an integrand becomes infinite before beginning to evaluate an integral.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $\int_1^\infty(3/x^2)dx = \lim_{b\to\infty}(-3/x)\Big|_1^b = \lim_{b\to\infty}(3 - 3/b) = 3$.

5. Using the method of partial fractions, $\int_2^\infty[dx/(x^2 - 1)] = (1/2)\int_2^\infty[dx/(x - 1) - dx/(x + 1)] = \lim_{b\to\infty}(1/2)\ln|(x - 1)/(x + 1)|\Big|_2^b = (1/2) \times \{\lim_{b\to\infty}\ln[(b - 1)/(b + 1)] + \ln 3\}$. By l'Hôpital's rule, $\lim_{b\to\infty}\ln[(b - 1)/(b + 1)] = \lim_{b\to\infty}\ln(1) = 0$. Thus, $\int_2^\infty[dx/(x^2 - 1)] = \ln 3/2$.

9. We want to find $g(x)$ so that $|f(x)| < g(x)$ and that $\int_a^\infty g(x)dx$ converges. If this is possible, then the comparison test states that $\int_a^\infty f(x)dx$ also converges. $f(x) = 1/(3 + x^3) < 1/x^3 = g_1(x)$ on $[1,\infty)$ and $f(x) < 1/3 = g_2(x)$ on $[0,1]$. Thus, $\int_0^\infty[dx/(3 + x^3)] = \int_0^1[dx/(3 + x^3)] + \int_1^\infty[dx/(3 + x^3)] < \int_0^1(dx/3) + \int_1^\infty(dx/x^3) = (x/3)\Big|_0^1 + \lim_{b\to\infty}(-1/2x^2)\Big|_1^b = 1/3 + 1/2 - \lim_{b\to\infty}(1/2b^2) = 5/6$. Thus, $\int_0^\infty[dx/(3 + x^3)]$ converges.

13. Use the method of Example 6. Find $g(x)$ less than the given $f(x)$ and then, show that $g(x)$ diverges. Choose $g_1(x) = 1/2 < 1/\sqrt{2 + x^2} = f(x)$ on $[0,1]$ and choose $g_2(x) = 1/x = 1/\sqrt{x^2} < f(x)$ on $[1,\infty)$. Thus, $\int_0^\infty\left(dx/\sqrt{2 + x^2}\right) = \int_0^1\left(dx/\sqrt{2 + x^2}\right) + \int_1^\infty\left(dx/\sqrt{2 + x^2}\right) > \int_0^1(dx/2) + \int_1^\infty(dx/x) = (x/2)\Big|_0^1 + \lim_{b\to\infty}\ln x\Big|_1^b$. Since $\lim_{b\to\infty}\ln b$ diverges, $\int_0^\infty\left(dx/\sqrt{2 + x^2}\right)$ diverges.

17. $\int_0^{10}(dx/x^{2/3}) = \lim_{a\to0+}\int_a^{10}(dx/x^{2/3}) = \lim_{a\to0+} 3x^{1/3}\Big|_a^{10} = 3\lim_{a\to0+}\left(\sqrt[3]{10} - \sqrt[3]{a}\right) = 3\sqrt[3]{10}$.

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21. The integral is improper at $x = 0$, so $\int_{-1}^{1}[dx/(x^2 + x)] = \int_{-1}^{0}[dx/(x^2 + x)] + \int_{0}^{1}[dx/(x^2 + x)]$. Since $x^2 \leq x$ on $(0,1)$, we have $\int_{0}^{1}[dx/(x^2 + x)] \geq \int_{0}^{1}(dx/2x) = \lim_{a\to 0+}\int_{a}^{1}(dx/2x) = \lim_{a\to 0+}(\ln x/2)\Big|_{a}^{1}$, which approaches $-\infty$. Since part of the original integral diverges, the entire integral diverges.

25. $\tan^{-1}x$ lies between $-\pi/2$ and $\pi/2$ for all x, so we can apply the comparison test with $f(x) = \tan^{-1}x/(2 + x)^3$ and $g(x) = (\pi/2)/(2 + x)^3$. $\int_{-1}^{\infty}g(x)dx = (\pi/2)[\lim_{b\to\infty}\int_{1}^{b}(2 + x)^{-3}dx] = (\pi/2)\lim_{b\to\infty}-(2 + x)^{-2}/2\Big|_{1}^{b} = (\pi/2)(-1/2 + \lim_{b\to\infty}[1/2(2 + b)^2]) = (\pi/2)(-1/2) = -\pi/4$. Since $\int_{-1}^{\infty}g(x)dx$ converges, so does $\int_{-1}^{\infty}f(x)dx$.

29. Let $f(x) = 1/(5x^2 + 1)^{2/3}$ and $g(x) = 1/(5x^2)^{2/3} = 1/\sqrt[3]{25}x^{4/3}$. $\int_{1}^{\infty}|f(x)|dx \leq \int_{1}^{\infty}g(x)dx = (1/\sqrt[3]{25})\int_{1}^{\infty}(dx/x^{4/3}) = \lim_{b\to\infty}(-3/x^{1/3})\Big|_{1}^{b} = 3$. Thus, $\int_{1}^{\infty}f(x)dx$ converges by the comparison test.

33. Note that $\int_{-\infty}^{-2}(1/x^{6/5} - 1/x^{4/3})dx = -\int_{\infty}^{2}[1/(-u)^{6/5} - 1/(-u)^{4/3}]du = \int_{2}^{\infty}[1/(-u)^{6/5} - 1/(-u)^{4/3}]du = \lim_{b\to\infty}\{\int_{2}^{b}[(-u)^{-6/5} - (-u)^{-4/3}]du\}$. Since each exponent is less than -1, by the result of Example 2, both terms converge. Thus, $\int_{-\infty}^{-2}(1/x^{6/5} - 1/x^{4/3})dx$ converges.

37. The integrand is unbounded at 0, so $\int_{-\infty}^{2}(1/x^{5/3} - 1/x^{4/3})dx = \lim_{a\to-\infty}\lim_{b\to 0-}\int_{a}^{b}(1/x^{5/3} - 1/x^{4/3})dx + \lim_{c\to 0+}\int_{c}^{2}(1/x^{5/3} - 1/x^{4/3})dx = \lim_{a\to-\infty}\lim_{b\to 0-}(-3/2x^{2/3} + 3/x^{1/3})\Big|_{a}^{b} + \lim_{c\to 0+}(-3/2x^{2/3} + 3/x^{1/3})\Big|_{c}^{2}$. This diverges toward $-\infty$, so the integral is divergent.

41. The parametric equations $\theta = t$, $r = t^{-k}$ are equivalent to $r = \theta^{-k}$, so $r'(\theta) = -k\theta^{-k-1}$. From Section 10.5, the arc length in polar coordinates is $L = \int_{\pi/2}^{\infty}\sqrt{(-k\theta^{-k-1})^2 + (\theta^{-k})^2}\,d\theta = \int_{\pi/2}^{\infty}(\sqrt{k^2 + \theta^2}/\sqrt{\theta^{2k+2}})d\theta \geq \int_{\pi/2}^{\infty}(\sqrt{\theta^2}/\theta^{k+1})d\theta = \int_{\pi/2}^{\infty}(d\theta/\theta^k)$. Since $\theta > 1$, we have $\int_{\pi/2}^{\infty}(\sqrt{k^2 + \theta^2}/\sqrt{\theta^{2k+2}})d\theta \leq \int_{\pi/2}^{\infty}(\sqrt{k^2\theta^2 + \theta^2}/\theta^{k+1})d\theta = \sqrt{k^2 + 1}\int_{\pi/2}^{\infty}(\theta/\theta^{k+1})d\theta = \sqrt{k^2 + 1} \times \int_{\pi/2}^{\infty}(d\theta/\theta^k)$. Thus, we have $\sqrt{k^2 + 1}\int_{\pi/2}^{\infty}(d\theta/\theta^k) \geq L \geq \int_{\pi/2}^{\infty}(d\theta/\theta^k)$, so

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41. (continued)

L acts like $\int_{\pi/2}^{\infty}(d\theta/\theta^k)$. By a method similar to that of Example 2, this converges for $k > 1$.

The case $k = 0$ is special because the spiral self-intersects forming a circle of radius 1 . Thus, the arc length is finite for $k = 0$ or $k > 1$.

45. In Example 11, Luke travels 10 million miles. In this problem, he must wake up before he gets within 1 million miles of the sun, so the distance travelled is 9 million miles. Thus, the time to travel 9 million miles is $[(9 \times 10^6)/(\sqrt{3}/2)](1/10^6) = 6\sqrt{3}$ hours ≈ 10.39 hours.

49. The limit is $\lim_{A\to 0+}[\int_{-3}^{-A}(dx/x) + \int_{A}^{2}(dx/x)]$. Recall that an improper integral converges only if each part separately converges. Since $\lim_{A\to 0+}\int_{A}^{2}(dx/x) = \lim_{A\to 0+}\ln|x|\,|_{A}^{2}$ diverges, the entire integral diverges.

53. (a) If $x = (\tau - \mu)/\sigma$, then $x = x_1$ when $\tau = a$ and x approaches $-\infty$ as τ approaches $-\infty$. $dx = d\tau/\sigma$, so substitution yields $p = \int_{-\infty}^{x_1}(1/\sqrt{2\pi}\sigma)e^{-x^2/2}(\sigma dx) = \int_{-\infty}^{x_1}(1/\sqrt{2\pi})e^{-x^2/2}dx$.

(b) If $|x| > 1$, then $e^{-x^2/2} < e^{-x/2}$ and if $|x| < 1$, then $e^{-x^2/2} < 1$. Since the integrand is symmetric to the y-axis, $\int_{-\infty}^{\infty}e^{-x^2/2}dx = 2\int_{0}^{\infty}e^{-x^2/2}dx < 2\int_{0}^{1}dx + 2\int_{1}^{\infty}e^{-x/2}dx = 2 + 2\lim_{b\to\infty}\int_{1}^{b}e^{-x/2}dx = 2 + \lim_{b\to\infty}(-2e^{-x/2})\,|_{1}^{b} = 2 - 2(\lim_{b\to\infty}e^{-b/2} + 2/\sqrt{e}) = 2 + 4/\sqrt{e} < \infty$. Thus, the comparison test tells us that $\int_{-\infty}^{\infty}e^{-x^2/2}dx < \infty$.

57. If $0 < f'(x) < 1/x^2$ on $[0,\infty)$, the same is true on $[a,\infty)$ for $a > 1$. By the comparison test, we have $\int_{a}^{\infty}0\,dx < \int_{a}^{\infty}f'(x)dx < \int_{a}^{\infty}(dx/x^2)$, i.e., $0 < \lim_{b\to\infty}f(x)\,|_{a}^{b} < \lim_{b\to\infty}(-1/x)\,|_{a}^{b}$ by the fundamental theorem of calculus. Thus, $0 < \lim_{b\to\infty}f(b) - f(a) < 1/a$, i.e., $f(a) < \lim_{b\to\infty}f(b) < 1/a + f(a)$. Since $f'(a)$ exists, $f(a)$ also exists. Now, since $f'(x) > 0$, f is an increasing function. Therefore, $f(x)$ is a strictly increasing

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57. (continued)

function and $\lim_{b\to\infty} f(b)$ is bounded above by $1/a + f(a)$. Thus, the limit exists.

SECTION QUIZ

1. Evaluate the following improper integrals, if they converge:

   (a) $\int_{-\infty}^{-3}[x/(x^2 - 1)]\,dx$

   (b) $\int_{-\infty}^{10} e^{4x}dx$

   (c) $\int_0^3 x^2 \ln x\,dx$

2. The integral $\int_{-10}^{10}[(x - 3)/(x + 5)^2]\,dx$ can be evaluated by substituting $u = x + 5$ to get $\int_{-5}^{15}[(u - 8)/u^2]\,du = (\ln|u| + 8/u)\Big|_{-5}^{15} = \ln 3 + 8/15 + 8/5$. Review these calculations and explain the errors, if any.

3. Discuss the convergence or divergence of the following integrals:

   (a) $\int_0^{50}[dx/(x^4 + x^3 + x^2 + 1)]$

   (b) $\int_0^{\infty}[dx/(x^{3/2} + 1)]$

   (c) $\int_{-2}^{1}[x^3/(x^4 - 1)]\,dx$

4. Ever since old man Anderson passed away 25 years ago, no one has set foot in his mansion. Each year, on the 13th of March, neighbors have reported strange, screeching noises from inside the mansion. This year, Nasty Nicholas wanted to prove his manliness by spending the night in the mansion. On the morning of the 14th, Nasty Nicholas emerged shaking with fear. One hour later, his hair began turning white at a rate of $25x^2/(x^4 + x)$, in strands of hair per hour. Will all of Nasty Nick's hair eventually turn white due to the big scare? Explain.

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ANSWERS TO PREREQUISITE QUIZ

1. (a) $+\infty$

   (b) $+\infty$

   (c) $+\infty$

2. (a) $\ln|x| + C$

   (b) $\sin x + x^3/3 + C$

   (c) $xe^x - e^x + C$

ANSWERS TO SECTION QUIZ

1. (a) Diverges

   (b) $e^{40}/4$

   (c) $9 \ln 3 - 3$

2. The integral is improper at $x = -5$, so $\int_{-10}^{10}[(x - 3)/(x + 5)^2]\,dx = \int_{-10}^{-5}[(x - 3)/(x + 5)^2]\,dx + \int_{-5}^{10}[(x - 3)/(x + 5)^2]\,dx$. Then substituting $u = x + 5$ gives $\int_{-5}^{0}[(u - 8)/u^2]\,du + \int_{0}^{15}[(u - 8)/u^2]\,du$, which diverges.

3. (a) Converges; it is not even improper on $[0,50]$.

   (b) Converges; it is less than $\int_0^1 dx + \int_1^\infty (dx/x^{3/2})$.

   (c) Diverges, $\int_{-2}^{-1}|x^3/(x^4 - 1)|\,dx > \int_{-2}^{-1}|(x^4 - 1)^{-1}|\,dx$.

4. By comparison, $\int_1^\infty [25x^2/(x^4 + x)]\,dx \leq \int_1^\infty (25/x^2)dx = (-25/x)\big|_1^\infty = 25$. Thus, the big scare will cause less than 25 strands of hair to turn white.

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11.4 Limits of Sequences and Newton's Method

PREREQUISITES

1. Recall how to compute a limit by using basic properties (Sections 1.2 and 11.1).
2. Recall how to compute a limit by using l'Hôpital's rule (Section 11.2).
3. Recall the formula used for the linear approximations (Section 1.6).

PREREQUISITE QUIZ

1. Compute $\lim_{x\to 2}(2/x - x)$.
2. Compute $\lim_{x\to\infty}(2/x)$.
3. Compute $\lim_{x\to\pi}[(\cos x + 1)/(\pi^2 - \pi x)]$.
4. If f is differentiable, write down the linear approximation to $f(x + \Delta x)$.

GOALS

1. Be able to find the limit of a sequence.
2. Be able to find roots by using Newton's method.

STUDY HINTS

1. Limit of sequences. A sequence is merely a list of numbers. For almost all sequences that we will consider, a definite pattern, which may be expressible by a formula, will exist. A limit ℓ exists if for any $\varepsilon > 0$, a_n is within $(\ell - \varepsilon, \ell + \varepsilon)$ whenever n is large enough. Example 3 shows how to use the precise definition.
2. Limit of "functional" sequences. If a sequence a_n is given by a formula, i.e., $a_n = f(n)$ for a function $f(x)$, then the limit of the sequence as n approaches ∞ is the same as the limit of the function

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2. (continued)

as x approaches ∞. Notice that since n is restricted to be an integer, a sequence may have a limit even though the function does not have a limit. For example, the sequence $\lim_{n\to\infty} \sin \pi n$ has a limit, but $\lim_{x\to\infty} \sin \pi x$ does not have a limit.

3. Limit of powers. The box located at the bottom of p. 542 is obvious if you recall the discussion about exponents in Chapter 6. Alternatively, you may remember simple examples: 2^n gets large with n, while $(1/2)^n$ gets small.

4. Comparison test. Notice how this statement is very similar to the comparison test of Section 11.1. Again, there is a "sandwich" effect.

5. Newton's method. This procedure uses the linear approximation. You should either memorize the iteration formula, which is $x_{n+1} = x_n - f(x_n)/f'(x_n)$, or learn to derive it. Since we are looking for a root, we seek an approximation so that $f(x_n + \Delta x) \approx f(x_n) + f'(x_n)\Delta x = 0$. Thus, rearrangement yields $\Delta x = -f(x_n)/f'(x_n)$, and so $x_{n+1} = x_n + \Delta x = x_n - f(x_n)/f'(x_n)$. The formula is used repeatedly until $x_{n+1} \approx x_n$, i.e., $\Delta x \approx 0$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This exercise is analogous to Example 1. $10a_n = 1 + 1/10 + \ldots + 1/10^{n-1}$, so $10a_n - a_n = 9a_n = 1 - 1/10^n$, i.e., $a_n = 1/9 - (1/9)(1/10^n)$, i.e., $1/9 - a_n = (1/9)(1/10^n)$, which we want to be less than $1/10^6$. Thus, n must be at least 6.

5. The sequence is determined by substituting $n = 0, 1, 2, \ldots$ into $k_n = n^2 - 2\sqrt{n}$. Hence, $k_0 = (0)^2 - 2\sqrt{0} = 0$; $k_1 = (1)^2 - 2\sqrt{1} = -1$; $k_2 = (2)^2 - 2\sqrt{2} = 4 - 2\sqrt{2}$; $k_3 = (3)^2 - 2\sqrt{3} = 9 - 2\sqrt{3}$; $k_4 = (4)^2 -$

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5. (continued)

$2\sqrt{4} = 12$; $k_5 = (5)^2 - 2\sqrt{5} = 25 - 2\sqrt{5}$. Thus, the sequence is 0 , -1 , $4 - 2\sqrt{2}$, $9 - 2\sqrt{3}$, 12 , $25 - 2\sqrt{5}$, ...

9. The first term a_0 is given as $1/2$. The next term is $a_1 = a_{0+1} = [1/(0 + 1)]\Sigma_{i=0}^{0}(1/2) = (1/1)(1/2) = 1/2$; $a_2 = a_{1+1} = (1/2)(a_0 + a_1) = (1/2)(1) = 1/2$; $a_3 = a_{2+1} = (1/3)(3/2) = 1/2$; $a_4 = a_{3+1} = (1/4)(2) = 1/2$; $a_5 = a_{4+1} = (1/5)(5/2) = 1/2$. Thus, the sequence is $1/2$ for all n .

13. We need to choose N such that $|a_n - \ell| < \varepsilon$ for all $n \geqslant N$. Note that if $3/2n < \varepsilon$, then $3/(2n + 1) < \varepsilon$ and $1/n < 2\varepsilon/3$. If we choose $N \geqslant 3/2\varepsilon$, then for $n > N$, we have $|3/(2n + 1) - 0| < 3/2n < 3/2N < (3/2)/(3/2\varepsilon) = \varepsilon$.

17. Divide by n^2/n^2 to get $\lim_{n\to\infty}[(1/n - 3)/(1 + 1/n^2)] = (0 - 3)/(1 - 0) = -3$.

21. Note that $(\sin n)^2 \leqslant 1$ for all n and $\lim_{n\to\infty}[1/(n + 2)] = \lim_{n\to\infty}[(1/n)/(1 + 2/n)] = 0$; therefore, the absolute value of the limit is less than or equal to $\lim_{n\to\infty}[1/(n + 2)] = 0$. Thus, we conclude that the limit must be 0 .

25. For $a_n = \sqrt[n]{n/2}$, we calculate $a_{100} \approx 1.03999$, $a_{1000} \approx 1.00623$, and $a_{1000000} \approx 1.00001$. Thus, we guess that $\lim_{n\to\infty} a_n = 1$. By l'Hôpital's rule, $\lim_{x\to\infty}(x/2)^{1/x} = \exp[\lim_{x\to\infty}(\ln(x/2)/x)] = \exp[\lim_{x\to\infty}(1/2x)/1] = e^0 = 1$.

29. We use the fact that $\lim_{n\to\infty} r^n = 0$ if $0 \leqslant r < 1$. Here $r = 1/8$, so $\lim_{n\to\infty}(1/8)^n = 0$.

33. The limit is $\{[\lim_{n\to\infty} 3b + \lim_{n\to\infty}(1/4)^n]/[\lim_{n\to\infty}(n^2 - 1)]\}^3 = \lim_{n\to\infty}(3b/(n^2 - 1))$. Divide by $(n^2/n^2)^3$ to get $\lim_{n\to\infty}[(3b/n^2)/(1 - 1/n^2)]^3 = (0/1)^3 = 0$.

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37. The iteration formula is $x_{n+1} = x_n - (x_n^5 + x_n^2 - 3)/(5x_n^4 + 2x_n)$. Starting with $x_0 = 0$, we get $x_1 = 0 - (-3)/0$, so this method does not work. Starting with $x_0 = 2$, we get $x_1 = 1.607143$; $x_2 = 1.325370$; $x_3 = 1.167936$; $x_4 = 1.121778$; $x_5 = 1.118357$; $x_6 = 1.118340$; $x_7 = 1.118340$. $x_6 = x_7$ to 6 decimal places, so one root is 1.118340 . The critical points are $\sqrt[3]{-2/5}$ and 0 , and sketching the graph reveals only one root, so our answer is 1.118340 .

41. Analysis of the graphs of $\tan x$ and αx shows that λ_1 is near $\pi/2$; λ_2 is near $3\pi/2$; λ_3 is near $5\pi/2$. The iteration formula we will be using is $x_{n+1} = x_n - (\tan x_n - \alpha x_n)/(\sec^2 x_n - \alpha)$. Since we want $\tan x_n - \alpha x_n = 0$, a good choice for x_0 is some number just larger than $\tan^{-1}(\alpha n\pi/2)$, where $n = 1$, 3 , and 5 .

For $\alpha = 2$, we start with $x_0 = 1.263$, then $x_3 = 1.1655820$ and $x_4 = 1.1655612$. Starting with $x_0 = 4.607$, we have $x_2 = 4.6042168$ and $x_3 = 4.6042168$. Starting with $x_0 = 7.791$, we get $x_2 = 7.7898996$ and $x_3 = 7.7898838$.

For $\alpha = 3$, we start with $x_0 = 1.362$, then $x_3 = 1.3241947$ and $x_4 = 1.3241945$. Starting with $x_0 = 4.642$, we have $x_2 = 4.6406836$ and $x_3 = 4.6406836$. Starting with $x_0 = 7.812$, we get $x_2 = 7.8113345$ and $x_3 = 7.8113345$.

For $\alpha = 5$, we start with $x_0 = 1.444$, then $x_2 = 1.432042$ and $x_3 = 1.4320322$. Starting with $x_0 = 4.670$, we have $x_1 = 4.6695888$ and $x_2 = 4.6695848$. Starting with $x_0 = 7.829$, we get $x_2 = 7.8284393$ and $x_3 = 7.8284393$.

(continued on next page)

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41. (continued)

λ \ α	2	3	5
λ_1	1.1656	1.3242	1.4320
λ_2	4.6042	4.6407	4.6696
λ_3	7.7899	7.8113	7.8284

45. The first term is $a_1 = 2^1$. The second term is $a_2 = 2^2$; the third is $(2)^2 = 2^4$; $a_4 = (2^4)^2 = 2^8$; $a_5 = (2^8)^2 = 2^{16}$. Note that the exponent of 2 is doubling each time we press the "x^2" key, so $a_n = 2^{(2^{n-1})}$.

49. After 1 half-life, only half of the substance is left. After 2 half-lives, half of the half, or 1/4 is left. In general, after n half-lives, $1/2^n$ is left; therefore, the sequence, beginning with a_0 , is 1 , 1/2 , 1/4 , 1/8 , 1/16 , ... , $1/2^n$. $\lim_{n\to\infty} a_n = 0$ taking $r = 1/2$ in the limits of powers.

53. By the sum rule for limits, $\lim_{n\to\infty}[3n/(4n + 1) + (-1)^n \sin n/(n + 1)] = \lim_{n\to\infty}[3n/(4n + 1)] + \lim_{n\to\infty}[(-1)^n \sin n/(n + 1)]$. Dividing by n/n yields $\lim_{n\to\infty}[3/(4 + 1/n)] + \lim_{n\to\infty}[(-1)^n(\sin n/n)/(1 + 1/n)] = 3/4 + \lim_{n\to\infty}(-1)^n(\sin n/n)$. Since $|(-1)^n \sin n| < 1$, $\lim_{n\to\infty}[(-1)^n \sin n/n] = 0$. Thus, the limit is 3/4 .

57. (a) Suppose $\lim_{n\to\infty} b_n < L$, then $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n < L - L = 0$. This says that $\lim_{n\to\infty}(b_n - a_n) < 0$, but the hypothesis that $a_n < b_n$ implies $\lim_{n\to\infty}(b_n - a_n) > 0$, so we have a contradiction. Thus, the original assumption is false, and it must be the case that $\lim_{n\to\infty} b_n \geq L$.

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57. (b) According to the definition of the limit, $\lim_{n\to\infty} a_n = L$ means that $|a_n - L| < \varepsilon$ or $-\varepsilon + L < a_n < \varepsilon + L$ whenever $n > N_1$. Likewise, $\lim_{n\to\infty} c_n = L$ means that $-\varepsilon + L < c_n < \varepsilon + L$ whenever $n > N_2$. By hypothesis, $-\varepsilon + L < a_n < b_n < c_n < \varepsilon + L$ or $|b_n - L| < \varepsilon$ whenever $n > N_3 = \max(N_1, N_2)$. Thus, we have $\lim_{n\to\infty} b_n = L$.

SECTION QUIZ

1. Use Newton's method to approximate a root of $f(x) = x^3/3 - x^2 - 1$ to three decimal places. Start with $x_0 = -3$. If your calculator is programmable, try starting with $x_0 = 1$. (Hint: 30 steps are required if $x_0 = 1$ on a TI-58C. Your calculator may be different.)
2. Using Newton's method, find the solution of $y^5 + y^3 + y = 2$ accurate to two decimal places.
3. (a) If $f(x) = \cos^2 x\pi$ is a function, what is $\lim_{x\to\infty} f(x)$?
   (b) If $a_n = \cos^2 n\pi$ is a sequence, what is $\lim_{n\to\infty} a_n$?
   (c) Comment about your answers to parts (a) and (b).
4. Find the limit of the following sequences if they exist:
   (a) $\lim_{n\to\infty}[(-1)^{2n} n/(n^2 - n + 5)]$
   (b) $\lim_{n\to\infty}[(2n^3 - n + 1)/(3n^3 - 50n^2)]$
   (c) $\lim_{n\to\infty}[((-1)^n \sin(\pi n/4))/(n^2 + 2)]$
5. Bouncing Bobby, the aspiring four-year-old trampolinist, was practicing on his parent's bed. The ceiling is eight feet above the floor. The bed is 24 inches off the ground. On his n^{th} jump, Bouncing Bobby's height above the floor is $96 - (1/2)^n(72)$ in inches.
   (a) How high does Bobby bounce up to after each of his first six jumps?

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5. (b) If Bouncing Bobby continues to jump for a long time before his parents discover his joyful playing, how high will he be jumping when he is discovered?

ANSWERS TO PREREQUISITE QUIZ

1. -1
2. 0
3. 0
4. $f(x + \Delta x) \approx f(x) + [f'(x)]\Delta x$

ANSWERS TO SECTION QUIZ

1. 3.279
2. 0.87
3. (a) Does not exist.
   (b) 1
   (c) The variable for sequences only takes on integer values, so the limit of a sequence may exist even if the limit of the corresponding function does not exist.
4. (a) 0
   (b) 2/3
   (c) 0
5. (a) 60 ; 78 ; 81 ; 91.5 ; 93.75 ; and 94.875 inches
   (b) 96 inches

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11.5 Numerical Integration

PREREQUISITES

1. Recall how to express an integral as a Riemann sum (Section 4.3).
2. Recall the relationship between integrals and areas (Section 4.2 and 4.6).

PREREQUISITE QUIZ

1. Using 10 intervals and $f(c_i)$ where c_i is the midpoint of each interval, approximate $\int_0^1 x^2\,dx$ as a Riemann sum. Express your answer using summation notation.
2. What is the relationship between $\int f(x)dx$ and the area under $f(x)$?

GOALS

1. Be able to integrate numerically by using Riemann sums, the trapezoidal rule, or Simpson's rule.
2. Be able to estimate the errors incurred by using Riemann sums, the trapezoidal rule, or Simpson's rule.

STUDY HINTS

1. Riemann sums. After division into equal subintervals, evaluate at one chosen point in each interval. Add up each evaluation and then multiply by $(b - a)/n$, the length of a subinterval. Often, the integrand is evaluated at either the right endpoint or the left endpoint.
2. Riemann error. The error made by using Riemann sums depends on the first derivative of the integrand and the length of the subintervals to the first power.

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3. Trapezoidal rule. In this method, the interval of integration is divided into n equal parts. At each division point, the integrand is evaluated and counted twice except at the endpoints. This is then multiplied by $(b - a)/2n$, one-half of a subinterval's length.

4. Trapezoidal error. The error in the trapezoidal rule estimate depends on the second derivative of the integrand and the square of the subinterval's length.

5. Simpson's rule. The method requires an even number of subdivisions. Notice the pattern of the coefficients at the evaluated points; it is $1, 4, 2, 4, 2, \ldots, 2, 4, 1$. 2 and 4 alternate, while the endpoints are considered only once. The weighted sum is then multiplied by $(b - a)/3n$, one-third of a subinterval's length.

6. Simpson's error. The error in the estimate made by Simpson's rule depends on the fourth derivative of the integrand and the fourth power of the length of the subintervals.

7. Equal subdivisions unnecessary. Until this point, the methods presented have used equal subdivisions as a convenience for deriving the special formulas. Be aware that none of the numerical integration methods presented here require equal subdivisions. However, Simpson's method does require an even number of intervals and that the length of the i^{th} interval equal the length of the $(i + 1)^{th}$ interval for odd i.

8. Error estimates. Most beginning calculus courses will not require memorization of the error estimates; however, you should consult your instructor.

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SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. For $\int_{-1}^{1}(x^2 + 1)dx$, with $n = 10$, we have $(b - a)/n = 1/5$ and $x_i = -1 + i/5$. Thus, the Riemann sum is: $(1/5)\Sigma_{i=1}^{10}(x_i^2 + 1) = (1/5)\Sigma_{i=1}^{10}[(-1 + i/5)^2 + 1] = [(1/5)\Sigma_{i=1}^{10}(1 - 2i/5 + i^2/25)] + 2 = 4 - (2/25)\Sigma_{i=1}^{10} i + (1/125)\Sigma_{i=1}^{10} i^2 = 4 - (2/25)[10(11)/2] + (1/125) \times [10(11)(21)/6] = 2.68$. The actual value of $\int_{-1}^{1}(x^2 + 1)dx = (x^3/3 + x)\Big|_{-1}^{1} = 8/3 \approx 2.6667$.

5. With $n = 10$, Simpson's rule tells us that $\int_0^1 \left(x/\sqrt{x^3 + 2}\right)dx \approx (1/30)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_8) + 4f(x_9) + f(x_{10})]$, where $f(x) = x/\sqrt{x^3 + 2}$ and $x_i = i/10$. Thus, the integral is approximately $(1/30)[0/\sqrt{2} + 0.4/\sqrt{2.001} + 0.4/\sqrt{2.008} + 1.2/\sqrt{2.027} + 0.8/\sqrt{2.064} + 2/\sqrt{2.125} + 1.2/\sqrt{2.216} + 2.8/\sqrt{2.343} + 1.6/\sqrt{2.512} + 3.6/\sqrt{2.729} + 1/\sqrt{3}] \approx 0.3246$.

9. Simpson's rule needs adjacent equidistant intervals, so let $\Delta x_1 = 0.2$ in $[0,1.2]$ and let $\Delta x_2 = 0.1$ in $[1.2,1.8]$. $n = 6$ for both subdivisions fulfilling the requirement that n be even. Write the integral as a sum of two integrals and apply Simpson's rule to each. $\int_0^{1.8} f(x)dx = \int_0^{1.2} f(x)dx + \int_{1.2}^{1.8} f(x)dx = (1.2/18)[2.037 + 4(1.980) + 2(1.843) + 4(1.372) + 2(1.196) + 4(0.977) + 0.685] + (0.6/18)[0.685 + 4(0.819) + 2(1.026) + 4(0.799) + 2(0.662) + 4(0.538) + 0.555] = (0.2/3)(26.116) + (0.1/3)(13.24) = 2.1824$.

13. If $f(x) = x + \sin x$, then $f'(x) = 1 + \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f^{iv}(x) = \sin x$. The maximum error from the trapezoidal rule is $[(b - a)/12]M_2(\Delta x)^2$, where M_2 is the maximum of $|f''(x)|$ in $[a,b]$. We have $M_2 = 1$ and $b - a = \pi/2$, so we want $(\pi/24)(\Delta x)^2 \leq 10^{-5}$ or $\Delta x \leq (10^{-5} \cdot 24/\pi)^{1/2} \approx 0.00874$. Therefore, we need $n = (\pi/2)/\Delta x \geq 180$. (Continued next page.)

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13. (continued)

For Simpson's rule, the maximum error is $[(b - a)/180]M_4(\Delta x)^4$, where M_4 is the maximum of $|f^{iv}(x)|$ in $[a,b]$. We have $M_4 = 1$ and $b - a = \pi/2$, so we want $(\pi/360)(\Delta x)^4 \leq 10^{-5}$ or $\Delta x \leq (10^{-5}\cdot 360/\pi)^{1/4} \approx 0.1840$. Therefore, we need $n = (\pi/2)/\Delta x \geq 9$.

17. The maximum error in the trapezoidal rule is $[(b - a)/12]M_2(\Delta x)^2$ where M_2 is the maximum of $f''(x)$ on $[a,b]$. So the maximum error is $(1/12)M_2(1/10)^2 = M_2/1200$. $f'(x) = \left\{\left[\sqrt{(1 + x^2)^4 + (1 - x^2)^2} + x(4(1 + x^2)(2x) + 2(1 - x^2)(-2x))/2\sqrt{(1 + x^2)^4 + (1 - x^2)^2}\right](1 + x^2)^3 - x\sqrt{(1 + x^2)^4 + (1 - x^2)^2}(3)(1 + x^2)^2(2x)\right\}/(1 + x^2)^6 = \{[((1 + x^2)^4 + (1 - x^2)^2) + 2x^2 + 6x^4](1 + x^2) - 6x^2[(1 + x^2)^4 + (1 - x^2)^2]\}/(1 + x^2)^4\sqrt{(1 + x^2)^4 + (1 - x^2)^2} = (2 - 6x^2 + 5x^4 - 25x^6 - 19x^8 - 5x^{10})/(1 + x^2)^4\sqrt{(1 + x^2)^4 + (1 - x^2)^2}$.

$f''(x) = \left\{(-12x + 20x^3 - 150x^5 - 152x^7 - 50x^9)(1 + x^2)^4 \times \sqrt{(1 + x^2)^4 + (1 - x^2)^2} - (2 - 6x^2 + 5x^4 - 25x^6 - 19x^8 - 5x^{10})\left[(2x^2 + 6x^4)(1 + x^2)^4/\sqrt{(1 + x^2)^4 + (1 - x^2)^2} + \sqrt{(1 + x^2)^4 + (1 - x^2)^2}(4)(1 + x^2)^3 \times (2x)\right]\right\}/(1 + x^2)^8[(1 + x^2)^4 + (1 - x^2)^2] = (-12x + 20x^3 - 150x^5 - 152x^7 - 50x^9)[(1 + x^2)^4 + (1 - x^2)^2](1 + x^2) - (2 - 6x^2 + 5x^4 - 25x^6 - 19x^8 - 5x^{10})[(2x^2 + 6x^4)(1 + x^2) + ((1 + x^2)^4 + (1 + x^2)^2)8x]/\left[(1 + x^2)^5 \times \left(\sqrt{(1 + x^2)^4 + (1 - x^2)^2}\right)^3\right] = (-56x - 4x^2 + 56x^3 - 4x^4 - 488x^5 + 26x^6 - 264x^7 + 46x^8 - 1298x^9 + 208x^{10} - 1458x^{11} + 312x^{12} - 948x^{13} + 154x^{14} - 372x^{15} + 30x^{16} - 90x^{17} - 16x^{19})/(1 + x^2)^5\left(\sqrt{(1 + x^2)^4 + (1 - x^2)^2}\right)^3$.

$f''(1) = -2.03$; $f''(0) = 0$; $f''(1/2) = -2.29$; $f''(1/4) = -3.2$; $f''(3/4) = -2.48$. From this data, we can guess that $M_2 < 6$, so the error is $< 6/1200 = 0.005$. Therefore, the first 2 digits are correct.

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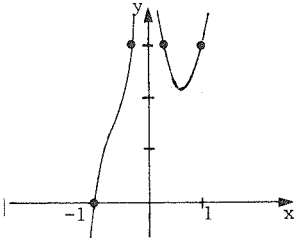
SECTION QUIZ

1. Use each of the methods of Riemann sums (evaluating at the right endpoint), the trapezoidal method, and Simpson's method to estimate $\int_0^1 f(y)dy$ from the given information. If not possible, state why.

   (a) $f(0) = 2$; $f(0.3) = 1$; $f(0.6) = -0.5$; $f(0.7) = -2$; $f(1) = 1$

   (b) $f(0) = 3$; $f(0.25) = 0$; $f(0.5) = -0.5$; $f(0.75) = 0.3$; $f(1) = 0$

   (c) $f(0) = -1$; $f(1/3) = -1$; $f(2/3) = 0$; $f(1) = 0.5$

2. 

For the figure at the left, $f(-1) = 0$; $f(-1/3) = 3$; $f(1/3) = 3$; and $f(1) = 3$. Can the trapezoidal rule be used to estimate the area under the curve on $[-1,1]$? If yes, estimate it. If no, explain why.

3. Santa Claus' elves produce $x^3 + x^2$ dollars worth of toys during the x^{th} week of the new working season. They begin work 12 weeks after Christmas and work continuously for 40 weeks. Thus, the total toy production in one year is $\int_0^{40}(x^3 + x^2)dx$.

   (a) Estimate toy production to the nearest cent by the trapezoidal rule with $n = 4$.

   (b) Estimate toy production to the nearest cent by Simpson's rule with $n = 4$.

   (c) What is the exact amount of toys produced in one year?

ANSWERS TO PREREQUISITE QUIZ

1. $\Sigma_{i=0}^{9}(2i + 1)^2/400$

2. They are equal.

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ANSWERS TO SECTION QUIZ

1. (a) Riemann, 0.25 ; Trapezoidal, 0.25 ; Simpson's can't be used since the third interval does not equal the fourth interval.

   (b) Riemann, -0.050 ; Trapezoidal, 0.325 ; Simpson's, $4/15 \approx 0.267$.

   (c) Riemann, $-1/6 \approx -0.167$; Trapezoidal, $-5/12 \approx -0.417$; Simpson's can't be used since there is an odd number of intervals.

2. Yes; 5

3. (a) $702,000.00

   (b) $661,333.33

   (c) $661,333.33

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11.R Review Exercises for Chapter 11

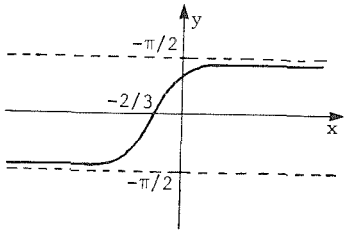
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the method of Example 1, Section 11.1. $f(x) - \ell = (x^2 + x - 1) - 1 = (x - 1 + 1)^2 + (x - 1 + 1) - 2 = (x - 1)^2 + 2(x - 1) + 1 + (x - 1) + 1 - 2 = (x - 1)^2 + 3(x - 1)$. By the properties of absolute value, $|f(x) - \ell| \leq |x - 1|^2 + 3|x - 1| = \delta^2 + 3\delta$. Now, assuming that $\delta < 1$, $|f(x) - \ell| \leq 4\delta < \varepsilon$. Therefore, if we choose δ to be the minimum of 1 or $\varepsilon/4$, then the definition of the limit is satisfied.

5. By the rational function and the composite function rules, $\lim_{x\to 0} \tan[(x + 1)/(x - 1)] = \tan(-1)$.

9. Multiply numerator and denominator by $\sqrt{2x^2 + 1} + \sqrt{2}x$ to get $\lim_{x\to\infty}\left[(2x^2 + 1 - 2x^2)/(\sqrt{2x^2 + 1} + \sqrt{2}x)\right] = \lim_{x\to\infty}\left[1/(\sqrt{2x^2 + 1} + \sqrt{2}x)\right]$. The denominator tends to ∞ as x tends to ∞ and the numerator is finite, so the entire limit is 0 .

13. $\lim_{x\to 0}[x \sin(3/x^2)] = \left[\lim_{x\to 0} x\right]\left[\lim_{x\to 0} \sin(3/x^2)\right]$. $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0} \sin(3/x^2)$ is finite, oscillating between -1 and 1 . Thus, the limit is 0 .

17. 

The horizontal asymptotes occur at $y = \lim_{x\to\pm\infty} f(x)$. $\lim_{x\to\infty} \tan^{-1}(3x + 2) = \pi/2$ and $\lim_{x\to-\infty} \tan^{-1}(3x + 2) = -\pi/2$, so $y = \pm\pi/2$ are the horizontal asymptotes. The x-intercept occurs where $0 = \tan^{-1}(3x + 2)$, i.e., $x = -2/3$. The y-intercept is $\tan^{-1}2$.

21. This has the form ∞/∞ , so l'Hôpital's rule gives $\lim_{x\to\infty}[(3x^2 + 8)/(12x^2 - 18x)] = \lim_{x\to\infty}[6x/(24x - 18)] = \lim_{x\to\infty}(6/24) = 1/4$.

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25. This has the form $0/0$, so l'Hôpital's rule gives $\lim_{x\to 0}\left[\left(\sqrt{x^2+9}-3\right)/\sin x\right] = \lim_{x\to 0}[(1/2)(x^2+9)^{-1/2}(2x)/\cos x] = 0$.

29. This has the form $0/0$, so l'Hôpital's rule gives us $\lim_{x\to 2}\{[\cos(x-2)-1]/3(x-2)^2\}$. We can apply l'Hôpital's rule again to get $\lim_{x\to 2}[-\sin(x-2)/6(x-2)]$. Applying the same rule gives us $\lim_{x\to 2}[-\cos(x-2)/6] = -1/6$.

33. The function $x \cot x$ does not have the appropriate form, so it must be transformed before l'Hôpital's rule can be used. Thus, $\lim_{x\to 0} x \cot x = \lim_{x\to 0}(x/\tan x)$ and has the form $0/0$; therefore, l'Hôpital's rule gives us $\lim_{x\to 0}(1/\sec^2 x) = 1$.

37. $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty}(x^2/e^x)$ and has the form ∞/∞; therefore, l'Hôpital's rule gives us $\lim_{x\to\infty}(2x/e^x)$. Applying l'Hôpital's rule again yields $\lim_{x\to\infty}(2/e^x) = 0$.

41. Find the limit of the logarithm and then exponentiate. $\lim_{x\to 0+}[\ln(1+\sin 2x)^{1/x}] = \lim_{x\to 0+}[(1/x)\ln(1+\sin 2x)] = \lim_{x\to 0+}[\ln(1+\sin 2x)/x]$, which has the form $0/0$. By l'Hôpital's rule, we get $\lim_{x\to 0+}[(2\cos 2x/(1+\sin 2x))/1] = 2$. Thus, $\lim_{x\to 0+}[(1+\sin 2x)^{1/x}] = e^2$.

45. The integral is $\int_1^\infty (1/x^2)dx = \lim_{b\to\infty}\int_1^b x^{-2}dx = \lim_{b\to\infty}(-1/x)\Big|_1^b = \lim_{b\to\infty}(-1/b) + 1 = 1$. Thus, the integral converges to 1.

49. Substitute $u = x - 1$, so $\int_1^2 (dx/\sqrt{x-1}) = \int_0^1 (du/\sqrt{u}) = \lim_{a\to 0+}\int_a^1 u^{-1/2}du = \lim_{a\to 0+} 2\sqrt{u}\Big|_a^1 = 2 - 0 = 2$. Thus, the integral converges to 2.

53. Integration by parts gives us $\int x \ln x\,dx = (\ln x)(x^2/2) - \int(x^2/2)(dx/x) = x^2\ln x/2 - \int x\,dx/2 = x^2\ln x/2 - x^2/4 + C$. Therefore, $\int_0^1 x\ln x\,dx = \lim_{a\to 0+}\int_a^1 x\ln x\,dx = \lim_{a\to 0+}(x^2\ln x/2 - x^2/4)\Big|_a^1$. $\lim_{a\to 0+} a^2\ln a/2 = (1/2)\lim_{a\to 0+}[\ln a/(1/a^2)] = (1/2)\lim_{a\to 0+}[(1/a)/(-1/a^3)] = (1/2)\lim_{a\to 0+}(-a^2) = 0$ by l'Hôpital's rule, so the integral converges to $(0 - 1/4) - (0 - 0) = -1/4$.

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57. From Section 9.1, the volume is $\pi\int_0^\infty (xe^{-x})^2 dx$. Integrate by parts as follows: $\int x^2e^{-2x}dx = x^2e^{-2x}/(-2) + \int xe^{-2x}dx = -x^2e^{-2x}/2 + xe^{-2x}/(-2) + \int e^{-2x}dx/2 = -x^2e^{-2x}/2 - xe^{-2x}/2 - e^{-2x}/4 + C$. By l'Hôpital's rule, $\lim_{x\to\infty}(-x^2/2e^{2x}) = \lim_{x\to\infty}(-2x/4e^{2x}) = \lim_{x\to\infty}(-2/8e^{2x}) = 0$. Also, $\lim_{x\to\infty}(-x/2e^{2x}) = \lim_{x\to\infty}(-1/4e^{2x}) = 0$, and $\lim_{x\to\infty}(-e^{-2x}/4) = 0$. Thus, $\pi\int_0^\infty (xe^{-x})^2 dx = \pi[\lim_{b\to\infty}(-x^2/2e^{2x} - x/2e^{2x} - 1/4e^{2x})]\,|_0^\infty = \pi/4$.

61. Find the limit of the logarithm and then exponentiate. $\lim_{n\to\infty}[n \ln(1 + 8/n)] = \lim_{n\to\infty}[\ln(1 + 8/n)/(1/n)]$. This has the form 0/0 and l'Hôpital's rule gives us $\lim_{n\to\infty}[((-8n^{-2})/(1 + 8/n))/(-n^{-2})] = \lim_{n\to\infty}[8/(1 + 8/n)] = 8$. Thus, the limit of the original sequence is e^8 .

65. Divide by n/n to get $\lim_{n\to\infty}[1/(1 + 2/n)] = 1/1 = 1$.

69. For any integer n , $\sin \pi n = 0$. Thus, l'Hôpital's rule gives us $\lim_{n\to\infty}(\sin \pi n/n^{-3}) = \lim_{n\to\infty}(\pi \cos \pi n/(-3n^{-4}))$. Now, for any integer n , $\cos \pi n = \pm 1$, so the sequence becomes $\pm(\pi/3)\lim_{n\to\infty} n^4$. Thus, the sequence oscillates between $+\infty$ and $-\infty$. Therefore, the limit does not exist.

73. By l'Hôpital's rule, $\lim_{n\to\infty} \sqrt[2n]{3n} = \exp[\lim_{n\to\infty}((\ln 3n)/2n)] = \exp[\lim_{n\to\infty}(1/n)/2] = e^0 = 1$.

77. Note that $f(-2) = -12$ and $f(-1) = 4$, so let $x_0 = -1$. We use the iteration formula $x_{n+1} = x_n - (x_n^3 - 3x_n^2 + 8)/(3x_n^2 - 6x_n)$ to get $x_1 = -1.44444$, $x_2 = -1.35916$, $x_3 = -1.35531$, $x_4 = -1.35530$, $x_5 = -1.35530$. Thus, one root is -1.35530 . By long division, $(x^3 - 3x^2 + 8)/(x + 1.35530) = x^2 - 4.35530x + 5.90274$. The quadratic equation gives us $(4.35530 \pm \sqrt{-4.64232})/2$. Therefore, the only root is $-1.35530\ldots$.

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81. We have $(b - a)/2n = 1/20$ and $f(x) = x^2/\sqrt{x^2 + 1}$. The approximation of $\int_2^3 \left[x^2/\sqrt{x^2 + 1}\right]dx$ by the trapezoidal rule is $(1/20)[f(2) + f(3) + 2\Sigma_{i=1}^{9} f(2 + i/10)] = (1/20)(46.39836) = 2.31992$.

85. For continuity, we must have $\lim_{x\to 0} f(x) = f(0) = \cos(0) = 1$. $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \cos x = 1$ and $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 1 = 1$; therefore, the function is continuous. For differentiability, $\lim_{\Delta x\to 0}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\}$ must exist. $f(x_0) = f(0) = 1$. If $x < 0$, then $f(x_0 + \Delta x) = f(\Delta x) = 1$ because $f(x) = 1$ if $x < 0$. Therefore, $\lim_{\Delta x\to 0-}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\} = \lim_{\Delta x\to 0-}[f'(x_0 + \Delta x)/1] = 0/1$ because l'Hôpital's rule applies and $f'(\Delta x) = (d/dx)(1) = 0$. If $\Delta x > 0$, $f(x_0 + \Delta x) = \cos(x_0 + \Delta x)$. Thus $\lim_{x\to 0+}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\} = \lim_{\Delta x\to 0+}[f'(x_0 + \Delta x)/1] = \lim_{\Delta x\to 0+}\sin(x_0 + \Delta x) = 0$ because l'Hôpital's rule applies again. Since both one-sided limits are equal, the function is both continuous and differentiable.

89. (a) The right-hand side has the form $0/0$, so by l'Hôpital's rule we get $f''(x_0) = \lim_{h\to 0}\{[f'(x_0 + h) - 0 + f'(x_0 - h)(-1)]/2h\}$. This limit also has the form $0/0$, so l'Hôpital's rule yields $\lim_{h\to 0}\{[f''(x_0 + h) - f''(x_0 - h)(-1)]/2 = [f''(x_0) + f''(x_0)]/2 = f''(x_0)$.

(b) By definition, $f'''(x_0) = \lim_{h\to 0}\{[f''(x_0 + h) - f''(x_0)]/h\}$. Substituting the formula from part (a) yields $f'''(x_0) = \lim_{h\to 0}\{[(f(x_0 + 2h) - 2f(x_0 + h) + f(x_0))/h^2 - (f(x_0 + h) - 2f(x_0) + f(x_0 - h))/h^2]/h\} = \lim_{h\to 0}\{[f(x_0 + 2h) - 3f(x_0 + h) + 3f(x_0) - f(x_0 - h)]/h^3\}$. This can be verified by applying l'Hôpital's rule three times.

93. S_n is the Riemann sum for $f(x) = x + x^2$, where $\Delta t_i = 1/n$ and the partition is $a = t_0 = 0$, $t_1 = 1/n$, $t_2 = 2/n$, ... , $t_n = b = 1$. c_i has been chosen to be t_i , so as $n \to \infty$, $\Delta t_i \to 0$ and $S_n \to \int_0^1 (x + x^2)dx = (x^2/2 + x^3/3)\Big|_0^1 = 1/2 + 1/3 = 5/6$.

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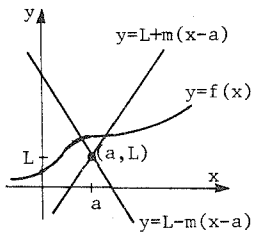
97. (a)



(b) $f(x)$ lies between the two lines means that $L - m(x - a) \leq f(x) \leq L + m(x - a)$. Subtracting L yields $-m(x - a) \leq f(x) - L \leq m(x - a)$, which implies $|f(x) - L| \leq m|x - a|$. Finally, division gives us $|[f(x) - L]/(x - a)| \leq m$.

(c) Given $\varepsilon > 0$, choose $\delta = \varepsilon/2m$. Then, the Lipschitz condition implies $|f(x) - L| \leq m|x - a| \leq mh = m(\varepsilon/2m) = \varepsilon/2 < \varepsilon$. Thus, we have shown that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| \leq \delta$, which is the definition of $\lim_{x \to a} f(x) = L$.

101. (a) If $f(x) = 0$ and $f'(x) \neq 0$, then $N(x) = x - f(x)/f'(x)$ becomes $N(x) = x - 0/f'(x) = x$. Now, if $N(x) = x$, then $N(x) = x - f(x)/f'(x)$ becomes $x = x - f(x)/f'(x)$, i.e., $0 = f(x)/f'(x)$, i.e., $0 = f(x)$.

(b) By the sum and quotient rules, $N'(x) = 1-[f'(x)f'(x) - f(x)f''(x)]/[f'(x)]^2$. This simplifies to $[(f'(x))^2 - (f'(x))^2 + f(x)f''(x)]/[f'(x)]^2 = f(x)f''(x)/[f'(x)]^2$.

(c) By the mean value theorem, we have $N'(\xi) = [N(x) - N(\bar{x})]/(x - \bar{x})$ for some ξ in $(x,\bar{x})$. Since $\bar{x}$ is a root of f, we have $N(\bar{x}) = \bar{x} - f(\bar{x})/f'(\bar{x}) = \bar{x} - 0/f'(\bar{x}) = \bar{x}$. Thus, $N'(\xi) = [N(x) - \bar{x}]/(x - \bar{x})$ and rearrangement yields $N(x) - \bar{x} = N'(\xi)[x - \bar{x}]$. Now, by part (b), $|N'(\xi)| = |f(\xi)f''(\xi)/(f'(\xi))^2| \leq |f(\xi)|M/p^2$.

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101. (c) (continued)

By Consequence 1 of the mean value theorem, $p \leq f'(x) \leq q$ implies $p \leq |[f(\xi) - f(\bar{x})]/(\xi - \bar{x})| \leq q$ or $p|\xi - \bar{x}| \leq |f(\xi)| \leq q|\xi - \bar{x}|$, since $f(\bar{x}) = 0$. And since $|x - \bar{x}| > |\xi - \bar{x}|$, we have $|f(\xi)| \leq q|x - \bar{x}|$. Therefore $|N'(\xi)| \leq q|x - \bar{x}|M/p^2$. Finally, $|N(x) - \bar{x}| = |N'(\xi)||x - \bar{x}| \leq (qM/p^2)|x - \bar{x}|^2$, so $C = qM/p^2$.

(d) Using the formula, $|N(x) - \bar{x}| \leq (qM/p^2)|x - \bar{x}|^2$, we substitute $p = 2.8$, $q = 3.0$, and $M = 2$ since $f'(x) = 2x$ and $f''(x) = 2$. In $[1.4,1.5]$, the maximum of $x - \bar{x}$ is 0.1, so $|N(x) - \bar{x}| \leq 0.00765$ for the first iteration. For the second iteration, $|x - \bar{x}| \leq 0.00765$, so $|N(x) - \bar{x}| \leq 0.000045$. For the third iteration, $|x - \bar{x}| \leq 0.000045$, so $|N(x) - \bar{x}| \leq 1.538 \times 10^{-9}$. For the fourth iteration $|x - \bar{x}| \leq 1.5 \times 10^{-9}$, so $|N(x) - \bar{x}| \leq 1.8 \times 10^{-18}$. For the fifth iteration, $|x - \bar{x}| \leq 1.8 \times 10^{-18}$, so $|N(x) - \bar{x}| \leq 2.5 \times 10^{-36}$. Thus, five iterations are needed to guarantee 20 decimal place accuracy.

TEST FOR CHAPTER 11

1. True or false.

(a) $\lim_{x \to \infty} \sin x = 0$.

(b) If Simpson's method is applicable, either the trapezoidal rule or the method of Riemann sums may also be used.

(c) If an interval is subdivided into an odd number of equal subintervals, Simpson's method may be used even if a discontinuity exists.

(d) The integral $\int_{-1}^{1}(dx/x^2) = -x^{-1}|_{-1}^{1}$ converges.

(e) By l'Hôpital's rule, $\lim_{x \to 0}[(x + 1)/\sin x] = \lim_{x \to 0}(1/\cos x)$.

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2. Let $f(x) = x(2x + 4)(x - 7)^2/(x + 3)(x - 7)(7x - 2)^2$.

   (a) Find the vertical and horizontal asymptotes of $f(x)$.

   (b) For the vertical asymptotes $x = x_0$, compute $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ for each x_0 found in part (a).

3. Use the $\varepsilon - \delta$ definition of the limit to prove that $\lim_{x \to -1} f(x) = 5$ if $f(x) = \begin{cases} x^2 - x + 3 & \text{if } x \neq -1 \\ 0 & \text{if } x = -1 \end{cases}$.

4. Compute the following limits:

   (a) $\lim_{x \to 0} (\tan x/x^3 - 1/x^2)$

   (b) $\lim_{x \to 0+} (1 + e^x)^x$

   (c) $\lim_{x \to \infty} e^{-x}\sqrt{x + 5}$

   (d) $\lim_{x \to 3} [(x + 3)/|\sqrt{x} - \sqrt{3}|]$

5. Evaluate the following improper integrals:

   (a) $\int_0^\infty xe^{-x}dx$

   (b) $\int_0^3 (dx/\sqrt[3]{x - 2})$

   (c) $\int_0^\infty [e^t/(1 + e^{2t})]\,dt$

6. Solve the equation $\sqrt{x} + x = 3$ accurate to three decimal places by using Newton's method.

7. Using the following methods with the partition (0 , 0.5 , 1.0 , 1.25 , 1.5 , 1.75 , 2.0) , approximate the arc length of $f(x) = \begin{cases} x^4 & \text{if } 0 \leq x \leq 1 \\ x^3 & \text{if } 1 \leq x \leq 2 \end{cases}$.

   (a) Use Simpson's method.

   (b) Use the trapezoidal method.

8. Do the following integrals converge or diverge? Justify your answer.

   (a) $\int_{-\infty}^\infty x \exp(-x^4)dx$

   (b) $\int_{-3}^2 [dx/(\sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x})]$

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8. (c) $\int_0^2 x(\ln x)^4 dx$

9. Functions of the form $\infty - \infty$ may have any value (if the limit exists). Give a simple example of a limit having the form $\infty - \infty$ whose value is:

   (a) $+\infty$

   (b) $-\infty$

   (c) 3 (Hint: consider $1/\ln x - 1/(x - 1)$.)

10. Krazy Karen's Kookie Kangaroos is a unique rent-a-kangaroo firm. People moving across town can rent a kangaroo, fill its pouch, and ride the kangaroos with their belongings across town. Krazy Karen, who used to work for a mad scientist, has developed energizing tablets for her kookie kangaroos. The average kangaroo can hop for $100n/(n + 3)$ kilometers after consuming n energizing tablets. The effects of the tablets last only one day.

   (a) Plot the kilometerage, km, vs. the number of tablets, n , for $n = 1 , 2 , 3 , 4 , 5 , 6$.

   (b) What is the limit of the sequence as $n \to \infty$.

   (c) Prove your answer in (b) by using the definition of the limit of sequences.

ANSWERS TO CHAPTER TEST

1. (a) False; limit doesn't exist.

   (b) True

   (c) False; Simpson's method requires an even number of subintervals.

   (d) False; $\int_{-1}^{1}(dx/x^2) = \int_{-1}^{0}(dx/x^2) + \int_{0}^{1}(dx/x^2) = (-x^{-1})\big|_{-1}^{0} + (-x^{-1})\big|_{0}^{1}$.

   (e) False; l'Hôpital's rule doesn't apply to functions with the form $1/0$.

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2. (a) Vertical asymptotes at $x = -3$ and $x = 2/7$; horizontal asymptotes at $y = 2/49$.

   (b) $\lim_{x \to -3-} f(x) = +\infty$; $\lim_{x \to -3+} f(x) = -\infty$; $\lim_{x \to 2/7-} f(x) = -\infty$; $\lim_{x \to 2/7+} f(x) = -\infty$.

3. Choose $\delta = \varepsilon/2$, so $|f(x) - L| = |x^2 - x + 3 - 5| = |(x + 1 - 1)^2 - (x + 1 - 1) - 2| < |x + 1|^2 + |x + 1| < \delta^2 + \delta < 2\delta$ if $\delta < 1$. Thus, if $|x + 1| < \delta$, then $|f(x) - L| < 2\delta = \varepsilon$.

4. (a) 1/3

   (b) 1

   (c) 0

   (d) $+\infty$

5. (a) 1

   (b) $(3/2)(1 - \sqrt[3]{4})$

   (c) $\pi/4$

6. 1.697

7. (a) 8.682

   (b) 8.955

8. (a) $\int_{-\infty}^{\infty} x \exp(-x^4)dx \leq \int_{-\infty}^{\infty} x \exp(-x^2)dx = \exp(-x^2)/2\Big|_{-\infty}^{\infty} = 0$. Therefore, $\int_{-\infty}^{\infty} x \exp(-x^4)dx$ converges.

   (b) The integral is improper around $x = 0$. If $-1 \leq x \leq 1$, then $\int_{-1}^{1} [dx/(\sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x})] \leq \int_{-1}^{1} [dx/3\sqrt[3]{x}] = \int_{-1}^{0} (dx/3\sqrt[3]{x}) + \int_{0}^{1} (dx/3\sqrt[3]{x}) = x^{2/3}/2\Big|_{-1}^{0} + x^{2/3}/2\Big|_{0}^{1} = 1$. Therefore, $\int_{-1}^{1} [dx/(\sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x})]$ converges.

   (c) The integral converges because $\lim_{x \to 0+} x(\ln x)^4 = 0$.

9. (a) $\lim_{x \to 0+} (2/x - 1/x)$

   (b) $\lim_{x \to 0+} (1/x - 2/x)$

   (c) $\lim_{x \to 1+} (6/\ln x - 6/(1 - x))$

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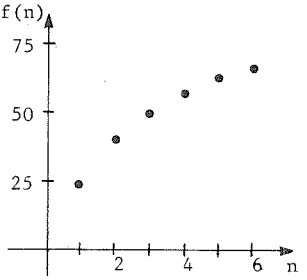
10. (a)



(b) 100

(c) Choose $N > (300 - 3\varepsilon)/\varepsilon$

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